Wagner de Campos Sanz*

A Criticism of Kripke's Semantic for Intuitionistic Logic

Uma crítica à semântica de Kripke para a lógica intuicionista

Resumo

Nesta breve nota, pretendemos examinar criticamente a semântica de Kripke para a lógica proposicional intuicionista. A semântica de Kripke é extensionalmente correta com relação à lógica proposicional intuicionista, isto é, o cálculo é correto e completo com respeito à semântica. O fragmento da lógica proposicional intuicionista contendo a disjunção e a implicação também é correto e completo com respeito às respectivas cláusulas semânticas. Entretanto, como procuraremos argumentar, a semântica de Kripke é intensionalmente enganadora, dado que a cláusula semântica de Kripke para a implicação é intensionalmente enganadora. Tal problema pode ser exemplificado quando consideramos o fragmento com disjunção e implicação.

Palavras-chave: semântica intuicionista, hipóteses, lógica proposicional

Abstract

This note aims to examine critically Kripke's semantics for propositional intuitionistic logic. Kripke's semantic is extensionally correct with respect to propositional intuitionistic logic, that is, the calculus is sound and complete with respect to the semantics. The fragment of propositional intuitionist logic containing disjunction and implication is also sound and complete with respect to the respective semantical clauses. However, we´ll argue, Kripke semantics is intensionally misleading, since Kripke's semantical implication clause is intensionally misleading. And the problem can be exemplified when we consider the fragment with disjunction and implication.

Keywords: intuitionistic semantics; hypotheses; propositional logic.

* Departamento de Filosofia, UFG.

1. Introduction

Kripke's semantic for intuitionistic logic came out in the 60's of the 20th century, see [Kri65]. Since then it has been regarded as one of the main semantical analyses of intuitionistic logic, if not from a philosophical point of view, at least from a practical point of view. It has been object of exposition in many different places, as in [Fit69].

However, other analyses have been proposed, notably those called proof-theoretical analyses. For a discussion concerning proof-theoretical definitions of validity and the problem of completeness see [Sch12], [Pie15] and [San16]. It is noteworthy that the problem we are going to expose seems to remain largely unnoticed in proof-theoretic semantics.

2. Intuitionistic Sequent Calculus

We will focus on the propositional fragment of intuitionistic logic containing implication and disjunction. Our considerations will be based on the intuitionistic sequent calculus. The operational rules considered involve only implication and disjunction. The inference rules will be presented linearly. Capital latin letters A, B and C represent sentences of the propositional language $\mathfrak{L}$. Small Latin letters a, b, c sentences belonging to $\mathfrak{L}_{At}$, the subset of atomic sentences of $\mathfrak{L}$. Capital Greek letters Γ and Δ represent finite multisets of sentences of $\mathfrak{L}$ (including the empty multiset).The fragment of propositional logic $LJM^{\rightarrow}$, contains the following structural rules:

Basic sequents: $A \vdash A$

Thinning on the left: $\Gamma \vdash A \Rightarrow \Gamma, B \vdash A$

Contraction on the left: $\Gamma, C, C, \Delta \vdash A \Rightarrow \Gamma, C, \Delta \vdash A$

Cut: $(\Gamma \vdash A \text{ and } A, \Delta \vdash B) \Rightarrow \Gamma, \Delta \vdash B$

$LJM^{\rightarrow}$ contains the following two operational rules for implication:

Implication introduction on the right: $\Gamma, A \vdash B \Rightarrow \Gamma \vdash A \rightarrow B$

Implication introduction on the left: $\Gamma \vdash A \text{ and } \Delta, B \vdash C \Rightarrow \Gamma, \Delta, A \rightarrow B \vdash C$

$LJM^{\rightarrow\vee}$ contains the following rules disjunction:

Disjunction introduction on the right: $\Gamma \vdash A \Rightarrow \Gamma \vdash A \vee B$ (L)

$\Gamma \vdash B \Rightarrow \Gamma \vdash A \vee B$ (R)

Disjunction introduction on the left: $\Gamma, A \vdash C \text{ and } \Delta, B \vdash C \Rightarrow \Gamma, \Delta, A \vee B \vdash C$

3. Kripke semantics for the fragments of LJM

A frame F in the intuitionistic Kripke semantics is an ordered pair <G, R>, where G is a set (frequently interpreted as a set of possible worlds or states of knowledge) and R is a partial order relation over the elements of G. A model M for the $LJM^{\to\vee}$ in Kripke semantics is given by a valuation over a frame F and two clauses: one for implication and another for disjunction. We use small Greek letters for elements of G. The valuation V_M is such that for every element δ of G and every sentence of $\mathfrak{L}_{At}$: V_M (δ, a) is either 0 (not forced) or 1 (forced). We also write $\delta\Vdash_M a$ when $V_M(\delta,a)=1$ and $\delta\nVdash_M a$ when $V_M(\delta,a)=0$. Additionally, V_M is such that, if $\delta R\gamma$ and $\delta\Vdash_M a$, then $\gamma\Vdash_M a$. It is clear from the above that for any sentence $a\in\mathfrak{L}_{At}$ and any $\delta\in G$: $\delta\Vdash_M a$ or $\delta\nVdash_M a$.
The clause for implication is:
$\delta\Vdash_M A\to B \Leftrightarrow$ for all γ such that δRγ $(\gamma\Vdash_M A \Rightarrow \gamma\Vdash_M B)$;
Which gives us a semantics for $LJM^{\to}$. The clause for disjunction is:
$\delta\Vdash_M A\vee B \Leftrightarrow \delta\Vdash_M A$ or $\gamma\Vdash_M B$.
Which added to the previous one gives us a semantics for $LJM^{\to\vee}$.
Lemma 1 (monotonicity) – For every sentence $A\in\mathfrak{L}$, if $\delta R\gamma$ and $\delta\Vdash_M A$, then $\gamma\Vdash_M A$.
Proof: straightforward induction on the complexity of A. QED

4. The distinguished model DM

Let the Frame F_{PA} be such that G_{PA} is the set of finite parts of $\mathfrak{L}_{At}$, i.e. any $\delta\in G_{PA}$ is finite and $\delta\subseteq\mathfrak{L}_{At}$. Let R_{PA} be ⊆. The Distinguished Model (DM) is a model over F_{PA} whose valuation V_{DM} is such that: for any $\delta\in G_{PA}$ and any $a\in\mathfrak{L}_{At}$, $\delta\Vdash_{DM} a\Leftrightarrow a\in\delta$. Starred small Greek letters will indicate either worlds of DM or simply finite sets of atomic sentences. Obviously, any finite set is a finite multiset.

As the empty world is an element of G_{PA} the following are instantiations of the clauses for DM:

$\Vdash_{DM} A\to B \Leftrightarrow$ for all γ* $(\gamma^*\Vdash_{DM} A \Rightarrow \gamma^*\Vdash_{DM} B)$ (DM.1)

$\delta^*\Vdash_{DM} A\vee B \Leftrightarrow \delta^*\Vdash_{DM} A$ or $\delta^*\Vdash_{DM} B$ (DM.2)

Of course, because of lemma 1, any sentence valid in the empty world is also valid for any world in DM.

Question 1: $\Vdash_{DM} (a\to(b\vee c))\to((a\to b)\vee(a\to c))$ for any three atomic sentences a, band c? The answer is yes.

Proof. According to (DM.1), $\Vdash_{DM}(a\to(b\vee c))\to((a\to b)\vee(a\to c)) \Leftrightarrow$ for all γ^* $[\gamma^*\Vdash_{DM}(a\to(b\vee c))\Rightarrow\gamma^*\Vdash_{DM}(a\to b)\vee(a\to c)]$. Suppose $\delta^*\Vdash_{DM}a\to(b\vee c)$. Thus, either $a\notin\delta^*$ or $a\in\delta^*$. Suppose $a\notin\delta^*$. If either $b=a$ or $c=a$, the result follows easily. If $b\neq a\neq c$, either $b\in\delta^*$ or $c\in\delta^*$, otherwise there would be an extension $\theta^*=\delta^*\cup\{a\}$ such that $\theta^*\nVdash_{DM} b\vee c$, contrary to our assumption. Hence, $\delta^*\Vdash_{DM}(a\to b)\vee(a\to c)$. Suppose $a\in\delta^*$, then either $b\in\delta^*$ or $c\in\delta^*$ according to our assumption and (DM.2), and the result is similar to above. Therefore, for all γ^* $[\gamma^*\Vdash_{DM}(a\to(b\vee c))\Rightarrow\gamma^*\Vdash_{DM}(a\to b)\vee(a\to c)]$. That is, $\Vdash_{DM}$ $(a\to(b\vee c))\to((a\to b)\vee(a\to c))$.

There is something quite curious here. For the empty world, i.e., that world in which all atomic sentences are not forced in DM, Mint's formula for any three atomic sentences holds semantically in that world and, consequently, in the model DM.

If we consider the pure calculus $LJM^{\to\vee}$, i.e. with an empty basis (with no basic atomic sequent), we can prove that $\vdash(a\to(b\vee c))\Rightarrow\vdash(a\to b)\vee(a\to c)$, but we cannot prove that $\vdash(a\to(b\vee c))\to((a\to b)\vee(a\to c))$.In other words, the rule is admissible but it is not derivable, for atomic sentences in general.

For sure, there is a Kripke model in which $\nVdash(a\to(b\vee c))\to((a\to b)\vee(a\to c))$. A three worlds model suffices to show that: (i)$\delta\nVdash a$, $\delta\Vdash a\to(b\vee c)$ and $\delta\nVdash(a\to b)\vee(a\to c)$; (ii)$\theta\supseteq\delta$, $\theta\Vdash a$, $\theta\Vdash b$; $\theta\Vdash a\to b$ and $\theta\Vdash b\vee c$; (iii) $\sigma\supseteq\delta$,$\sigma\Vdash a$, $\sigma\Vdash c$; $\sigma\Vdash a\to c$ and $\sigma\Vdash b\vee c$.

5. An important metaproperty of implication in $LJM^{\to}$ (but also in $LJM^{\to\vee}$)

Theorem 1 - $\Gamma\vdash A\to B \Leftrightarrow$ for all finite $\Delta\supseteq\Gamma$ $(\Delta\vdash A \Rightarrow \Delta\vdash B)$

Proof. From right to left. Suppose $\Gamma\vdash A\to B$. Suppose $\Lambda\supseteq\Gamma$. Suppose $\Lambda\vdash A$.The sequent $A\to B, A\vdash B$ is provable in $LJM^{\to}$. By cut $\Gamma, A\vdash B$. By cut again, $\Gamma,\Lambda\vdash B$. By contractions, $\Lambda\vdash B$.Therefore, for all finite $\Delta\supseteq\Gamma$ $(\Delta\vdash A \Rightarrow \Delta\vdash B)$.From left to right. Suppose for all finite $\Delta\supseteq\Gamma$ $(\Delta\vdash A \Rightarrow \Delta\vdash B)$. By instantiation, $\Gamma, A\vdash A \Rightarrow \Gamma, A\vdash B$. By basic sequent and thinning $\Gamma, A\vdash A$. Hence, $\Gamma, A\vdash B$. By implication introduction $\Gamma\vdash A\to B$. QED

The above property holds of implication independent of disjunction. It expresses the sufficient and necessary condition for having an implication on the right side of the sequent. The following expression also makes explicit the meaning of implication: $\Gamma \vdash A \rightarrow B \Leftrightarrow \Gamma, A \vdash B$. Actually, $\Gamma, A \vdash B \Leftrightarrow$ for all finite $\Delta \supseteq \Gamma$ ($\Delta \vdash A \Rightarrow \Delta \vdash B$). This is proved by using cut[1].

From our perspective the equivalence of theorem 1has to be taken as saying that Kripke semantics is intensionally misleading because the implication clause is misleading. As a particular case of the above theorem we have for the empty world of DM:$\vdash A \rightarrow B \Leftrightarrow$ for all Δ ($\Delta \vdash A \Rightarrow \Delta \vdash B$)[2]. While, of course, the following implication:

(i) If for all finite Δ ($\Delta \vdash A \Rightarrow \Delta \vdash B$), then for all (finite) γ^* ($\gamma^* \vdash A \Rightarrow \gamma^* \vdash B$)

is correct, the converse implication:

(ii) if for all (finite) γ^* ($\gamma^* \vdash A \Rightarrow \gamma^* \vdash B$), then for all finite Δ ($\Delta \vdash A \Rightarrow \Delta \vdash B$) is not correct.

The sentence (iii) bellow is a particular case of (ii). Let, a, b and c be distinct atomic sentences:

(iii) if for all γ^* [$\gamma^* \vdash a \rightarrow (b \vee c) \Rightarrow \gamma^* \vdash (a \rightarrow b) \vee (a \rightarrow c)$], then $a \rightarrow (b \vee c) \vdash a \rightarrow (b \vee c) \Rightarrow a \rightarrow (b \vee c) \vdash (a \rightarrow b) \vee (a \rightarrow c)$.

First, as γ^* is a finite set of atomic sentences, the supposition that $\gamma^* \vdash a \rightarrow (b \vee c)$ implies that there will be a cut-free derivation in which $\gamma^*, a \vdash b$ or $\gamma^*, a \vdash c$. Thus, $\gamma^* \vdash (a \rightarrow b) \vee (a \rightarrow c)$. Second, $a \rightarrow (b \vee c) \vdash a \rightarrow (b \vee c)$ is a basic sequent. Finally, $a \rightarrow (b \vee c) \vdash (a \rightarrow b) \vee (a \rightarrow c)$ is not derivable in $LJM^{\rightarrow \vee}$ with an empty basis.

6. Conclusions

From the extensional point of view Kripke's semantics and the intuitionistic propositional logic are equivalent, also in the fragment for implication. But, there is an important difference concerning the Distinguished Model. While the syntactical system does not derive $a \rightarrow (b \vee c) \vdash (a \rightarrow b) \vee (a \rightarrow c)$, DM validates it. Therefore, either we give preeminence to the syntactical system and reject the semantical characterization as a way intensionally incorrect of capturing the constructivist meaning of logical constants or we do the opposite.

From an historical point of view, the syntactical system, that is, Heyting's calculus came first in the 30´s. As we said, Kripke's semantic was published in the 60's. But of course, the question is not of temporal precedence in publication. It is a question of faithful representation. But then we should ask:

[1]Once cut is assumed, it can also be proved that $\Gamma, A \rightarrow B \vdash C \Leftrightarrow$ for all finite $\Delta \supseteq \Gamma$ ($\Delta, A \vdash B \Rightarrow \Delta \vdash C$) from $\Gamma \vdash A \rightarrow B \Leftrightarrow$ for all finite $\Delta \supseteq \Gamma$ ($\Delta \vdash A \Rightarrow \Delta \vdash B$), and vice-versa.

[2] Clearly, it is enough to take into account merely finite sets.

representation of what? Of the intuitionist position? Of a constructivist position among others?

Kripke semantics has been rejected from a more purist point of view, like Dummett's in LBM p. 26, since for these purists the metalevel use of third middle excluded for the relation of forcing would be truly unacceptable for a constructivist. We think that such a criticism is wrong headed. After all, a semantic must provide counter models, and in order to provide counter models we must be able to say when a sentence is not forced in a world. Kripke could as well just have laid down clauses for explaining when a sentence is not forced in a world, thus making no use of third middle.

However, we are inclined to assume the calculus as the constructivist basic standing point. The above sequent calculus takes into account a concept that has not received all the attention deserved in the recent history of logic. This is the concept of hypothesis. We do use hypotheses in our reasonings and the sequent calculus somehow captures what is to use a hypothesis. Kripke semantics, on the other hand, makes the notion of hypothesis dependent on the notion of state of knowledge or of possible world. This is what is involved in the quantification in Kripke's implication clause. And the DM model has in a sense all possible combinations of finite worlds. It is clear that the meta property of theorem 1 gives a criterion for introducing implication that is more restrictive. It requires quantification over, at least, all finite sets of hypotheses, and not only the atomic ones. These sets are defined by reference to a language and they do not require any epistemically unclear and dubious concept like that of state of knowledge.

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